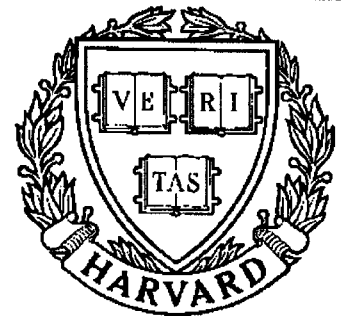


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## **The Dynamics of Coupled Planar Rigid Bodies Part II: Bifurcations, Periodic Solutions, and Chaos**

*by Y.-G. Oh, N. Sreenath,  
P.S. Krishnaprasad and J.E. Marsden*

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# ***The Dynamics of Coupled Planar Rigid Bodies***

## ***Part II: Bifurcations, Periodic Solutions, and Chaos***

Y.-G. Oh\*, N. Sreenath\*\*, P.S. Krishnaprasad\*\*, and J.E. Marsden\*

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## §1 Introduction

Part I of this paper, namely Sreenath, Oh, Krishnaprasad, and Marsden [1987], hereafter denoted [I], studied the Hamiltonian structure and equilibria for interconnected planar rigid bodies, with the primary focus being on the case of three bodies coupled with hinge joints. The Hamiltonian structure was obtained by the reduction technique, starting with the canonical Hamiltonian structure in material representation and then quotienting by the group of Euclidean motions. For three bodies, this Hamiltonian structure is as follows (see Figure 1): the phase space is  $P = S^1 \times S^1 \times \mathbb{R}^3$ , parametrized by the two joint angles  $\theta_{21} =: \phi$  and  $\theta_{32} =: \psi$  and three momenta  $\mu = (\mu_1, \mu_2, \mu_3)$  (conjugate to the three angular variables  $(\theta_1, \theta_2, \theta_3)$  for the three bodies) with the Poisson bracket

$$\begin{aligned} \{f, g\} = & \left( \frac{\partial f}{\partial \mu_1} - \frac{\partial f}{\partial \mu_2} \right) \frac{\partial g}{\partial \phi} - \left( \frac{\partial g}{\partial \mu_1} - \frac{\partial g}{\partial \mu_2} \right) \frac{\partial f}{\partial \phi} \\ & + \left( \frac{\partial f}{\partial \mu_2} - \frac{\partial f}{\partial \mu_3} \right) \frac{\partial g}{\partial \psi} - \left( \frac{\partial g}{\partial \mu_2} - \frac{\partial g}{\partial \mu_3} \right) \frac{\partial f}{\partial \psi} . \end{aligned} \quad (1.1)$$

This phase space is obtained by first reducing to center of mass coordinates and then getting rid of rotations via

$$P = \frac{T^*(S^1 \times S^1 \times S^1)}{S^1} \cong S^1 \times S^1 \times \mathbb{R}^3 .$$

Given a Hamiltonian  $H(\phi, \psi, \mu_1, \mu_2, \mu_3)$ , the evolution equations  $\dot{f} = \{f, H\}$  are equivalent to

$$\left. \begin{aligned} \dot{\mu}_1 &= \frac{\partial H}{\partial \phi} , \\ \dot{\mu}_2 &= \frac{\partial H}{\partial \psi} - \frac{\partial H}{\partial \phi} , \\ \dot{\mu}_3 &= -\frac{\partial H}{\partial \psi} , \\ \dot{\phi} &= \frac{\partial H}{\partial \mu_2} - \frac{\partial H}{\partial \mu_1} , \\ \dot{\psi} &= \frac{\partial H}{\partial \mu_3} - \frac{\partial H}{\partial \mu_2} . \end{aligned} \right\} \quad (1.2)$$

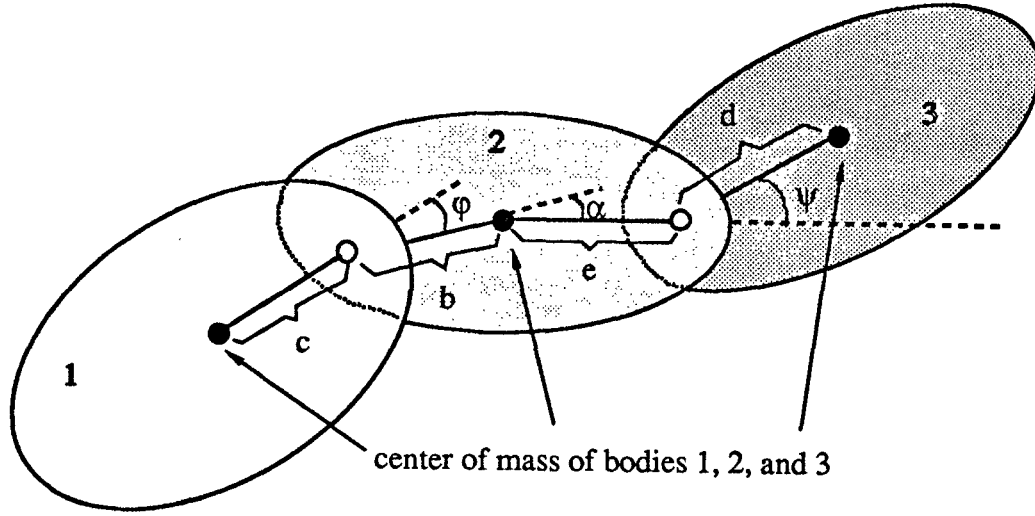


Figure 1

For this system the Hamiltonian is shown in [I] to be

$$H = \frac{1}{2} \langle \omega, J\omega \rangle = \frac{1}{2} \langle \mu, J^{-1}\mu \rangle \quad (1.3)$$

where

$$(\omega_1, \omega_2, \omega_3) = (\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3) = J_\mu$$

is the angular velocity vector and

$$J = \begin{bmatrix} \bar{I}_1 & \lambda_{12}(\phi) & \lambda_{31}(\phi + \psi) \\ \lambda_{12}(\phi) & \bar{I}_2 & \lambda_{23}(\psi) \\ \lambda_{31}(\phi + \psi) & \lambda_{23}(\psi) & \bar{I}_3 \end{bmatrix} \quad (1.4)$$

is the effective moment of inertia matrix; the entries are defined as follows: let  $c, b, e, d$  be the positive distances shown in Figure 1 (i.e., distances between the centers of mass and the hinge points; assume here that the center of mass of the central body is between the hinge points) and let  $\epsilon_{ij} = m_i m_j / (m_1 + m_2 + m_3)$ ,  $I_1, I_2, I_3$  be the planar moments of inertia of each body,

$$\bar{I}_1 = I_1 + (\epsilon_{12} + \epsilon_{13})c^2, \quad \bar{I}_3 = I_3 + (\epsilon_{23} + \epsilon_{13})d^2,$$

$$\tilde{I}_2 = I_2 + (\epsilon_{12} + \epsilon_{13})b^2 + (\epsilon_{23} + \epsilon_{13})e^2 + 2\epsilon_{13}be \cos \alpha$$

be the augmented moments of inertia, and

$$\lambda_{12}(\phi) = (\epsilon_{12} + \epsilon_{13})bc \cos \phi + \epsilon_{13} ce \cos(\phi + \alpha) ,$$

$$\lambda_{31}(\sigma) = \epsilon_{13}cd \cos \sigma ,$$

$$\lambda_{23}(\psi) = (\epsilon_{23} + \epsilon_{13})de \cos(\psi - \alpha) + \epsilon_{13}bd \cos \psi .$$

Equilibrium solutions are determined by setting the time derivatives in (1.2) to zero:

$$\left. \begin{aligned} \frac{\partial H}{\partial \phi} = \frac{\partial H}{\partial \psi} &= 0 , \\ \frac{\partial H}{\partial \mu_1} = \frac{\partial H}{\partial \mu_2} = \frac{\partial H}{\partial \mu_3} &= \omega_0 , \text{ a constant .} \end{aligned} \right\} \quad (1.5)$$

To further simplify the problem we will assume that the center of mass of the second body is aligned with the two hinge points; i.e., that  $\alpha = 0$  (see Figure 1). Then (1.5) is equivalent to the system

$$\left. \begin{aligned} \frac{\partial H}{\partial \mu_i} &= \omega_0 , \\ \sin(\phi + \psi) &= -\tau \sin \phi , \\ \sin \psi &= \kappa \sin \phi , \end{aligned} \right\} \quad (1.6)$$

where

$$\left. \begin{aligned} \kappa &= \frac{\epsilon_{13}(b+e)c + \epsilon_{12}bc}{\epsilon_{13}(b+e)d + \epsilon_{23}de} \\ \tau &= \frac{\epsilon_{13}(b+e) + \epsilon_{12}b}{\epsilon_{13}d} , \end{aligned} \right\} \quad (1.7)$$

and

as was shown in [I].

It was also shown in [I] that there are always four or six equilibria, amongst which are the four *fundamental equilibria*

$$(\phi, \psi) = (0, 0), (0, \pi), (\pi, 0), (\pi, \pi) . \quad (1.8)$$

When  $(\kappa, \tau)$  belongs to the shaded region in Figure 2, i.e.,  $|\tau - 1| < \kappa < \tau + 1$ , there are two other equilibria determined by

$$\cos \phi = \frac{1 - (\kappa^2 + \tau^2)}{2\kappa\tau} \quad \text{and} \quad \cos \psi = \frac{\kappa^2 - \tau^2 - 1}{2\tau} . \quad (1.9)$$

Correspondingly, the pairs  $(\phi, \psi)$  lie in the shaded region of Figure 3.

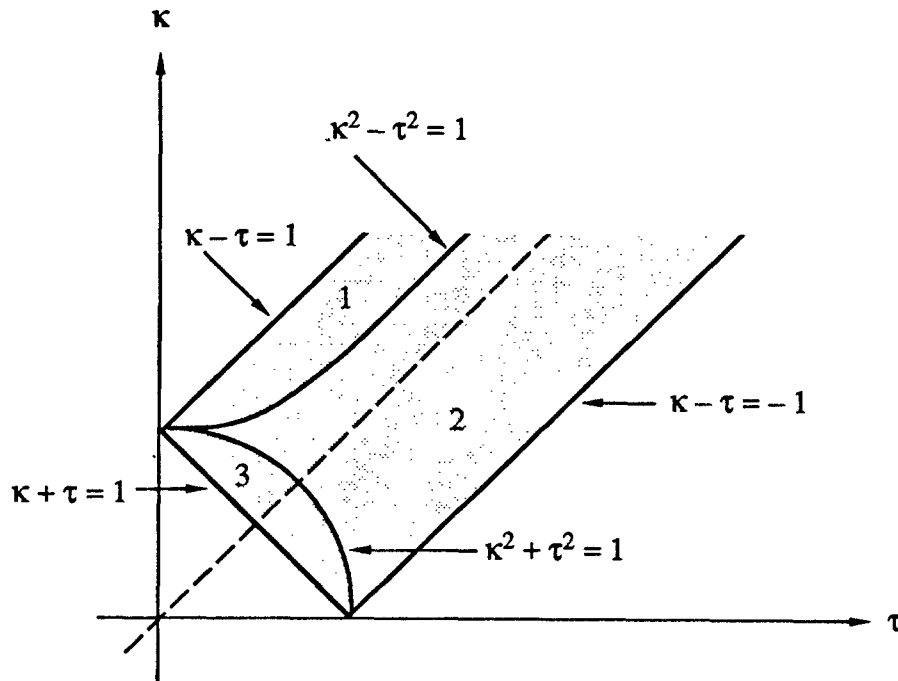


Figure 2

For example, if  $m_1 = m_2 = m_3$ ,  $b = e = \mu d$ , and  $c = \lambda d$ , then  $\kappa = \lambda$  and  $\tau = 3\mu$ , so the condition for four equilibria,  $|\tau - 1| < \kappa < \tau + 1$ , becomes  $|3\mu - 1| < \lambda < 3\mu + 1$ . For instance, if  $\mu = 1$ , this condition is  $2 < \lambda < 4$ . Thus as  $\lambda$  leaves the range  $[2, 4]$ , the number of solutions drops from six to four.

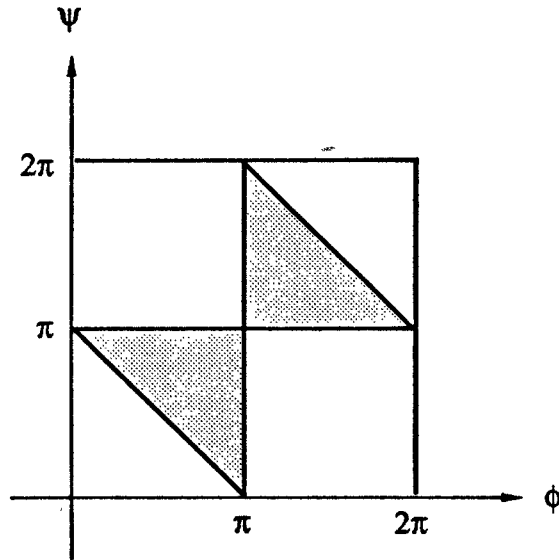


Figure 3

In [I] it was shown that the equilibrium  $(0, 0)$  representing the straight stretched out solution is stable for all system parameters. In §2 we study bifurcations of these equilibria and we determine the eigenvalue evolution of these bifurcations and thereby determine that the solutions that are not formally stable are not only unstable, but are spectrally and hence exponentially unstable, with non-zero eigenvalues of the linearized equations on the real axis. In §3 we use the version of the Weinstein-Moser theorem due to Montaldi, Roberts, and Stewart [1987] to show the existence of two families of periodic orbits (with symmetries) near the stable equilibrium  $(\phi, \psi) = (0, 0)$ ; they are shown to be spectrally stable when they are nonresonant. Finally, in §4 we show that the problem is, in general, non-integrable. This is done using the Melnikov method in the version given by Holmes and Marsden [1983] to show that the homoclinic orbit present in the integrable case  $d = 0$  leads to transverse homoclinic orbits for small  $d \neq 0$ . General conditions for integrability are not known to us.

We believe that the periodic solutions found in §4 are related to travelling waves in a long chain of  $n$  coupled bodies (with torsional springs) or in the corresponding continuum limit  $n \rightarrow \infty$ . This will be the subject of another investigation.



## §2 Bifurcation of Equilibria

In this section we relate the bifurcations of equilibria to the degeneracies of the Hessian of the energy function. This is used in the next section where we discuss the stability indices.

First of all, one can see directly from the equilibrium equations, as in [I], that a Hamiltonian pitchfork-type bifurcation occurs at each of the three unstable fundamental equilibrium solutions, as in Figure 4.

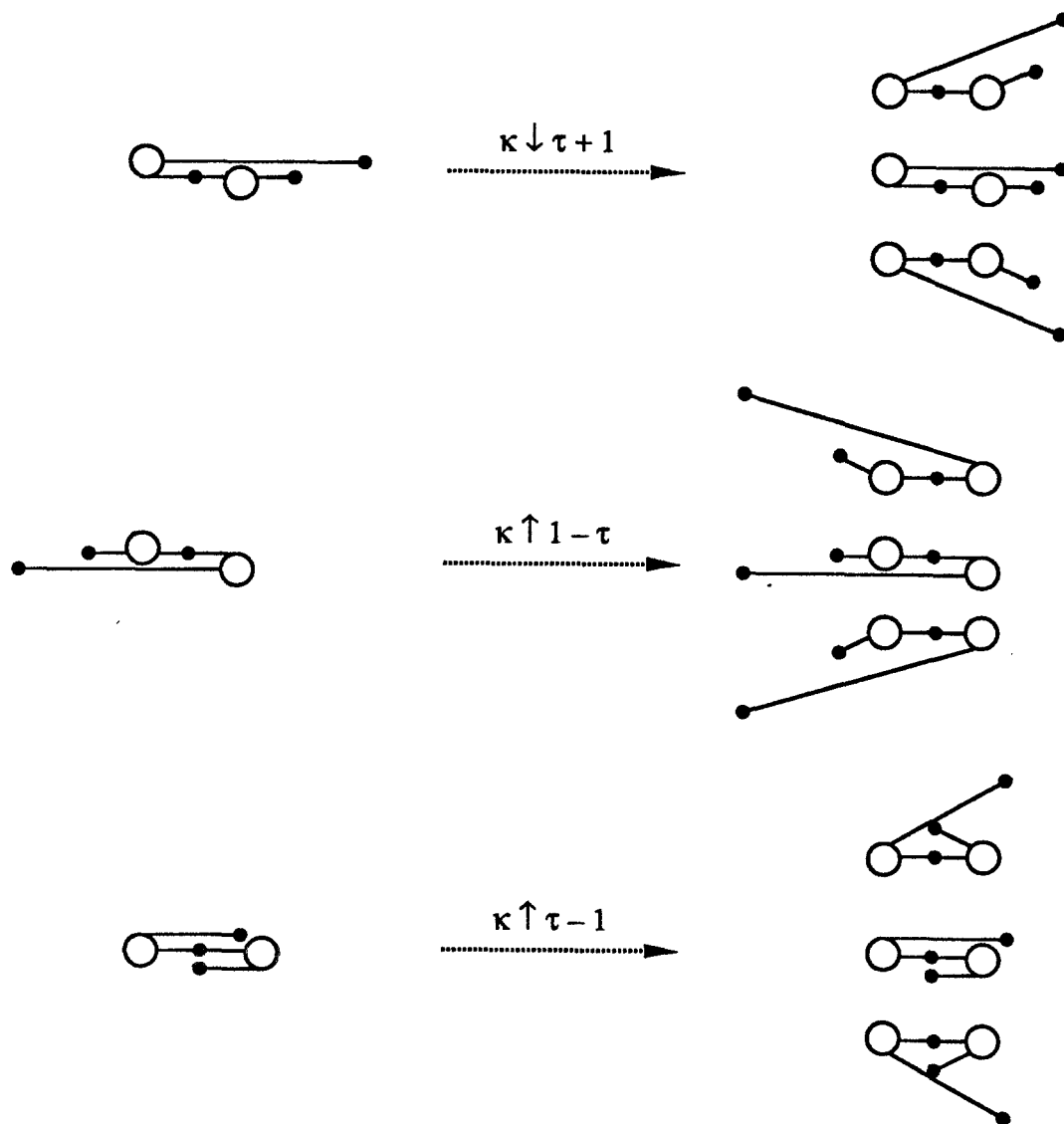


Figure 4

For example, if  $m_1 = m_2 = m_3$ ,  $c = \lambda d$ , and  $b = e = d$ , the evolution of equilibria as  $\lambda$  ranges from  $\lambda < 2$  to  $\lambda > 4$  is shown in Figure 5.

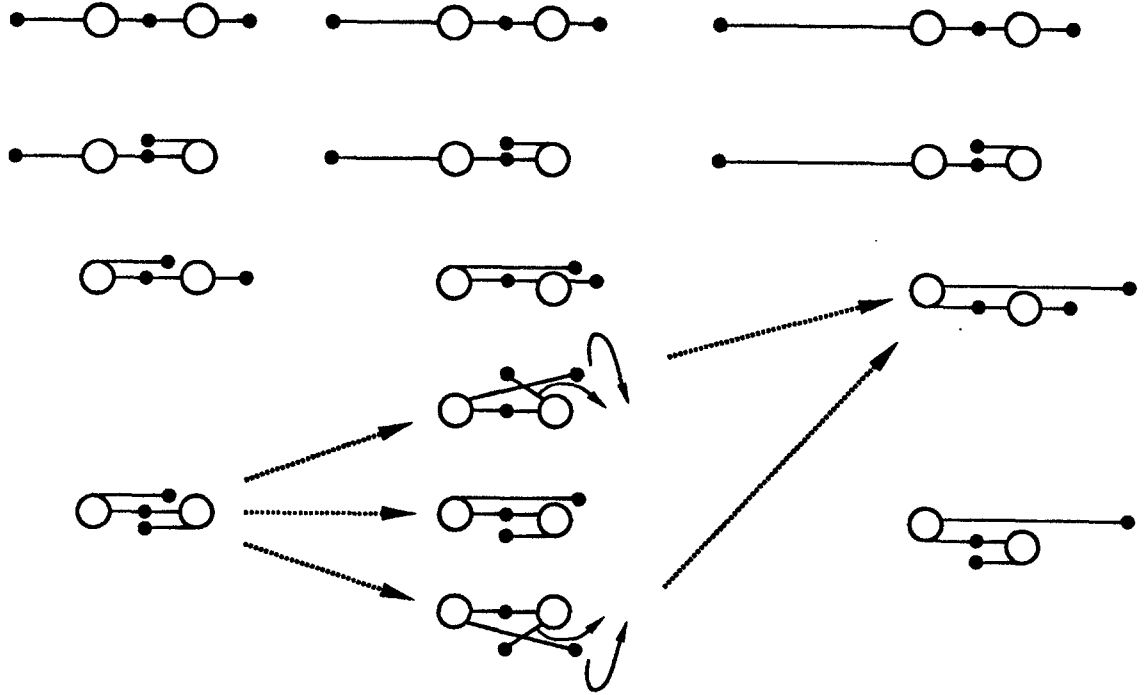


Figure 5

These bifurcations, which can all be seen by direct calculation, will now be related to the second variation, or Hessian, of the Hamiltonian. The symplectic leaves  $P_\mu$  of the phase space  $P$  are defined by setting  $\mu_1 + \mu_2 + \mu_3 = \mu$ , a constant. Equilibria for the system are exactly critical points of  $H_\mu$ , the restriction of  $H$  to  $P_\mu$ . At these points, the Hessian of  $H_\mu$  is simply the restriction of the second variation of  $H$  to tangent vectors of  $P_\mu$  at the equilibrium point in question. Since the Hamiltonian vector field restricted to the leaf has a zero eigenvalue iff the Hessian does, it is a priori clear that a bifurcation of equilibria occurs only if the Hessian along the leaf has a zero eigenvalue.

The Hessian is computed at one of the fundamental equilibria to have the form

$$\delta^2 H = \begin{bmatrix} J^{-1} & 0 \\ 0 & B \end{bmatrix} \quad (2.1)$$

as a  $5 \times 5$  matrix with the variables in the order  $(\mu_1, \mu_2, \mu_3, \phi, \psi)$  restricted to the subspace defined by  $\delta\mu_1 + \delta\mu_2 + \delta\mu_3 = 0$ , where  $J$  is given by (1.4) and where

$$B = (\text{positive constant}) \begin{bmatrix} \cos \phi + \frac{1}{\tau} \cos(\phi + \psi) & \frac{1}{\tau} \cos(\phi + \psi) \\ \frac{1}{\tau} \cos(\phi + \psi) & \frac{1}{\kappa} \cos \psi + \frac{1}{\tau} \cos(\phi + \psi) \end{bmatrix}. \quad (2.2)$$

Ignoring the positive constant, we note that at the fundamental equilibria,

$$\begin{aligned} B(0, 0) &= \begin{bmatrix} \frac{1}{\tau} + 1 & \frac{1}{\tau} \\ \frac{1}{\tau} & \frac{1}{\kappa} + \frac{1}{\tau} \end{bmatrix}, & B(\pi, 0) &= \begin{bmatrix} -\frac{1}{\tau} - 1 & -\frac{1}{\tau} \\ -\frac{1}{\tau} & \frac{1}{\kappa} - \frac{1}{\tau} \end{bmatrix}, \\ B(0, \pi) &= \begin{bmatrix} 1 - \frac{1}{\tau} & -\frac{1}{\tau} \\ -\frac{1}{\tau} & -\frac{1}{\kappa} - \frac{1}{\tau} \end{bmatrix}, & B(\pi, \pi) &= \begin{bmatrix} \frac{1}{\tau} - 1 & \frac{1}{\tau} \\ \frac{1}{\tau} & -\frac{1}{\kappa} + \frac{1}{\tau} \end{bmatrix}. \end{aligned} \quad (2.3)$$

Since  $J$  is positive definite, bifurcations of equilibria are determined by zero eigenvalues of  $B$ . Since

$$\det B(0, 0) = \frac{1}{\kappa} + \frac{1}{\tau} + \frac{1}{\tau\kappa} > 0,$$

$(0, 0)$  never bifurcates. Since

$$\det B(\pi, 0) = \frac{\kappa - \tau - 1}{\kappa\tau},$$

$$\det B(0, \pi) = \frac{1 - \kappa - \tau}{\kappa\tau},$$

and

$$\det B(\pi, \pi) = \frac{\tau - \kappa - 1}{\kappa\tau},$$

we can expect these equilibria to bifurcate at  $\kappa = \tau + 1$ ,  $\kappa = 1 - \tau$ , and  $\kappa = \tau - 1$ , respectively. As we saw above, this is confirmed by a direct analysis of the equilibria.

To analyze the stability of these equilibria notice first that the stretched out state  $(0, 0)$  is always stable, as we already know from [I]. For the state  $(\pi, \pi)$ , note that

$$\det B(\pi, \pi) = \frac{\tau - \kappa - 1}{\kappa\tau}, \quad \text{trace } B(\pi, \pi) = \frac{2\kappa - \tau\kappa - \tau}{\tau\kappa},$$

so  $B(\pi, \pi)$  has

- $\kappa > \tau - 1$  : one negative and one positive eigenvalue,
- $\kappa = \tau - 1$  : one negative and one zero eigenvalue,
- $\kappa < \tau - 1$  : two negative eigenvalues.

There are similar statements for the equilibria  $(\pi, 0)$  and  $(0, \pi)$  where  $\tau - 1$  is replaced by  $\tau + 1$  and  $1 - \tau$ , respectively.

**Theorem 2.1** *The equilibrium  $(0, 0)$  is always stable and the other fundamental equilibria are unstable, and in fact spectrally unstable.*

The proof relies on

**Lemma 2.2** *Let  $A$  and  $B$  be two real  $n \times n$  symmetric invertible matrices with different numbers of negative eigenvalues. Then the infinitesimally symplectic matrix*

$$\begin{bmatrix} 0 & B \\ -A & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

*has at least one positive (and so one negative) real eigenvalue.*

**Proof of Lemma 2.2**

$$\begin{aligned} \det \left( \lambda I_{2n} - \begin{bmatrix} 0 & B \\ -A & 0 \end{bmatrix} \right) &= \det \begin{bmatrix} \lambda I & -B \\ A & \lambda I \end{bmatrix} \\ &= \lambda^{2n} \det \begin{bmatrix} I & -\frac{1}{\lambda} B \\ \frac{1}{\lambda} A & I \end{bmatrix} \end{aligned}$$

(notice that  $\lambda \neq 0$  since we assume that  $A$  and  $B$  are invertible)

$$\begin{aligned}
 &= \lambda^{2n} \det \begin{pmatrix} I & -\frac{1}{\lambda} B \\ 0 & I + \frac{1}{\lambda^2} AB \end{pmatrix} \\
 &= \lambda^{2n} \det \left( I + \frac{1}{\lambda^2} AB \right) = \det (\lambda^2 + AB) .
 \end{aligned}$$

The lemma follows from this sublemma.

**Sublemma 2.3** *Under the same hypothesis as Lemma 2.2, the matrix  $AB$  has at least one negative eigenvalue.*

**Proof of Sublemma 2.3** Since we assume that  $A$  is invertible,  $A^{-1}$  has the same inertia index (number of negative eigenvalues) as  $A$ . Now, consider the 1-parameter family of symmetric matrices  $M(t) = tA^{-1} + (1-t)B$ ,  $0 \leq t \leq 1$ . We know that the set of invertible symmetric matrices has  $n+1$  components which are characterized by inertia indices. Since  $M(0)$  and  $M(1)$  have different indices and so are contained in different components and  $\{M(t)\}$  is connected, there must be some  $0 < t_0 < 1$  for which  $M(t_0)$  is not invertible, i.e., there exists a non-zero vector  $v$  such that  $M(t_0)v = 0$ , i.e.,

$$(t_0 A^{-1} + (1-t_0)B)v = 0 .$$

Multiplying by  $A$ , we get

$$ABv = -\frac{t_0}{1-t_0} v .$$

Here,  $-t_0/(1-t_0)$  is negative since  $0 < t_0 < 1$ . Therefore,  $AB$  has a negative eigenvalue. ■

**Remarks 1** We can refine this lemma to allow one of the matrices not to be invertible. More specifically, let us assume that  $A$  is invertible of the type  $(p, q)$  and  $B$  is of type  $(p', q', r)$  where  $r$  is the number of zero eigenvalues. Then  $AB$  must have at least one negative eigenvalue if  $p > p' + r$  or  $q > q' + r$ . This then yields Lemma 2.2 as before.

**2** A generalization of Lemma 2.2 was pointed out by J. Howard [1987]; similar criteria for Krein collisions (Hamiltonian Hopf bifurcations) would be of use in the case of *three* dimensional coupled rigid bodies (cf. Krishnaprasad, Grossman, and Marsden [1987]).

(3) Some related results and applications are given in Oh [1987].

**Lemma 2.3** *In the reduced symplectic space  $P_\mu$ , the Hessian of the reduced Hamiltonian has the form*

$$\begin{pmatrix} J' & 0 \\ 0 & B \end{pmatrix},$$

where  $J'$  is a positive definite  $2 \times 2$  matrix and  $B$  is the matrix (2.2) in the canonical coordinate near the equilibria given by  $(\phi, \psi, v_1, v_2)$ , where

$$v_1 = \frac{\mu_2 - \mu_1}{2}, \quad v_2 = \frac{\mu_3 - \mu_2}{2}.$$

**Proof of Lemma 2.3** This follows from the fact that at the equilibria,  $d^2H$  are given by (2.1) in  $T^*(S^1 \times S^1 \times S^1)/S^1$  which is parametrized by  $(\mu_1, \mu_2, \mu_3, \phi, \psi)$  and the fact that

$$\left( \frac{\mu_2 - \mu_1}{2}, \frac{\mu_3 - \mu_2}{2}, \phi, \psi \right)$$

are canonical coordinates on  $P_\mu$  near the equilibria; the latter is checked directly from the bracket (1.1) ■

**Proof of Theorem 2.1** We know that in canonical coordinates, the symplectic structure is given by

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and by Lemma 2.3, the Hessian has the form

$$\begin{pmatrix} J' & 0 \\ 0 & B \end{pmatrix}$$

where the number of negative eigenvalues of  $J'$  is zero and, as we illustrated at  $B(\pi, \pi)$ ,  $B$  has at least one negative eigenvalue. Therefore, by Lemma 2.2 the linearization of the Hamiltonian vector

least one negative eigenvalue. Therefore, by Lemma 2.2 the linearization of the Hamiltonian vector field at the equilibria, namely

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} J' & 0 \\ 0 & B \end{pmatrix},$$

has at least one real eigenvalue and so is spectrally unstable and so is non-linearly unstable. ■

Finally, we study the stability of the bifurcation branches at the equilibria. By Theorem 2.1, we know that the linearization of the Hamiltonian vector field at  $(\pi, \pi)$  (similarly for  $(0, \pi)$ ,  $(\pi, 0)$ ) has at least one and so two real eigenvalues. This came from observing that

$$B(\pi, \pi) = \begin{pmatrix} \frac{1}{\tau} - 1 & , & \frac{1}{\tau} \\ \frac{1}{\tau} & , & \frac{1}{\tau} - \frac{1}{k} \end{pmatrix}$$

has

$$\det B(\pi, \pi) > 0 \text{ if } \kappa < \tau - 1 ,$$

$$\det B(\pi, \pi) < 0 \text{ if } \kappa > \tau - 1 .$$

Using Lemma 2.2, we get Figure 6 for the positions of eigenvalues of  $DX_H(\pi, \pi)$  with respect to the parameters  $(\tau, \kappa)$ .

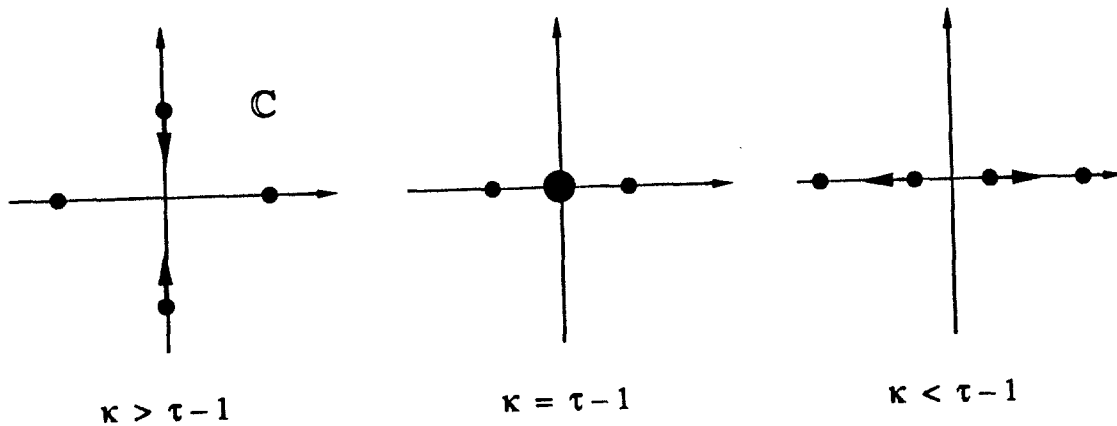


Figure 6

By the spectral property for the eigenvalues of an infinitesimally symplectic matrix, any small perturbation of the  $DX_H(\pi, \pi)$  at  $\kappa = \tau - 1$  must have real eigenvalues. Hence the  $DX_H$  at the bifurcated equilibria must have real eigenvalues at least near the bifurcation parameter. Thus we have the following theorem.

**Theorem 2.4** *All the bifurcation branches from unstable equilibria are themselves unstable.*



### §3 Periodic Solutions

We have seen that the straight stretched-out state  $(\phi, \psi) = (0, 0)$  is stable for all system parameters. Correspondingly, the second variation of the Hamiltonian on the symplectic leaf is positive definite at this point. If  $\omega_0$  denotes the angular velocity of this solution, its angular momentum is

$$\mu = \mu_1 + \mu_2 + \mu_3 = \omega_0 \sum_{ij=1}^3 J_{ij}(0, 0)$$

and its energy is

$$E_0 = \frac{1}{2} \langle \omega_0(1, 1, 1), \omega_0 J(1, 1, 1) \rangle = \frac{1}{2} \omega_0^2 \sum_{ij=1}^3 J_{ij}(0, 0) \quad (3.1)$$

where  $J$  is the matrix (1.4). The theorem of Weinstein [1973], [1978] and Moser [1976] gives the following:

**Theorem 3.1** *For any small  $\varepsilon > 0$ , there are at least two distinct periodic orbits near this equilibrium on the energy surface  $H^{-1}(E_0 + \varepsilon)$  in the leaf  $P_\mu$ .*

These periodic motions in the reduced symplectic manifold produce quasi-periodic motion on a torus in the original phase space  $T^*(S^1 \times S^1 \times S^1)$  by  $S^1$  symmetry.

Theorem 3.1 does not directly tell us properties of these periodic orbits, such as their spatial structure. We will obtain such information in the case of a symmetric system by applying an equivariant version of the Weinstein-Moser Theorem due to Montaldi, Roberts, and Stewart [1987].

We assume that the 3-body system is symmetric under the transformation of configuration space given by

$$(\theta_1, \theta_2, \theta_3) \mapsto (\theta_3, \theta_2, \theta_1) \text{ in } S^1 \times S^1 \times S^1. \quad (3.2)$$

This means, in effect, that bodies 1 and 3 are mechanically identical. This assumption gives the symmetries  $J_{11} = J_{33}$  and  $J_{12} = J_{23}$  of the metric  $J$  besides  $J$  being symmetric. The transformation (3.2) on  $S^1 \times S^1 \times S^1$  induces a  $\mathbb{Z}_2$ -action on the phase space  $T^*(S^1 \times S^1 \times S^1)$  and the Hamiltonian (= the kinetic energy) is invariant under this action; this is our symmetry assumption. Obviously, this  $\mathbb{Z}_2$ -action commutes with the diagonal  $S^1$ -action and so it induces a

symplectic  $\mathbb{Z}_2$ -action on the reduced space  $P_\mu$ . In canonical coordinates  $(\phi, \psi, v_1, v_2)$  on  $P_\mu$  where  $\phi = \theta_2 - \theta_1$ ,  $\psi = \theta_3 - \theta_2$ ,  $v_1 = (\mu_2 - \mu_1)/2$ , and  $v_2 = (\mu_3 - \mu_2)/2$ , this  $\mathbb{Z}_2$ -action can be written as

$$(\phi, \psi, v_1, v_2) \rightarrow (-\psi, -\phi, -v_2, -v_1) . \quad (3.3)$$

Its *fixed manifold* is given by

$$F = \{(\theta_{21}, \theta_{32}, v_1, v_2) \mid \phi + \psi = 0 \pmod{2\pi}, v_1 + v_2 = 0\} . \quad (3.4)$$

We have the following general facts about fixed manifold under symplectic actions:

**Proposition 3.2** *Let a compact Lie group  $G$  act symplectically on a symplectic manifold  $P$ . Then each component of the fixed point set  $\text{Fix}(G)$  is a symplectic submanifold of  $P$ .*

**Proof** See, for instance, Guillemin and Sternberg [1984]. ■

**Proposition 3.3** *Let  $H : P \rightarrow \mathbb{R}$  be a  $G$ -invariant Hamiltonian and let  $X_H$  be the associated Hamiltonian vector field. Then  $X_H$  is tangent to each component of  $\text{Fix}(G)$  and  $X_H|_{\text{Fix}(G)}$  has the Hamiltonian  $H|_{\text{Fix}(G)}$ .*

**Proof** See Golubitsky and Stewart [1987]. ■

From these propositions, we see that  $F$  is a symplectic submanifold (in fact, one can easily check this without referring to Proposition 3.2) and the restriction of the Hamiltonian  $H$  will give a Hamiltonian on  $F$ . Notice that  $F$  is actually diffeomorphic to  $T^*S^1$  which is two dimensional, and so the dynamics on it is completely integrable. From the general fact that zeros of  $X_H$  in  $F$  are zeroes of  $X_H|_F$ , we can see that  $X_H|_F$  has two zeroes and corresponding to  $(\phi, \psi) = (0, 0)$  and  $(\pi, \pi)$  among the four fundamental equilibria.

Focussing on the induced dynamics on  $F \cong T^*S^1$ , notice that any level surfaces of the reduced Hamiltonian in  $F$  is compact since those of the original Hamiltonian  $H$  are. Since we have proved that the induced Hamiltonian vector field  $X_H|_F$  has one stable equilibrium and one unstable one, the dynamics is qualitatively similar to the reduced dynamics of the coupled two body case. In particular, we conclude that the original Hamiltonian system in  $P_\mu$  has infinitely many periodic orbits and at least two homoclinic orbits.

We summarize the above discussions:

**Theorem 3.4** *For the symmetric coupled planar 3-body system, every symmetric initial condition gives rise to a symmetric periodic motion (up to diagonal action by  $S^1$ ) unless the initial energy is the same as the energy of the two equilibria  $(0, 0)$  and  $(\pi, \pi)$ . Moreover, the energy surface of the unstable equilibrium  $(\pi, \pi)$  contain two homoclinic orbits issuing from it.*

From this theorem we may expect that the slightly unsymmetric system will have chaotic phenomena. This may be proved by an adaptation of the Melnikov method; see §4 below.

We consider the dynamics on the energy surface  $H^{-1}(E_0 + \epsilon)$  for small  $\epsilon > 0$ . We already know from Theorem 3.1 that this level surface contains at least two distinct periodic orbits. To get information on their symmetry, we use the following:

**Theorem 3.5** (Equivariant Weinstein-Moser Theorem; Montaldi, Roberts, and Stewart [1987].) *Let  $G$  be a group acting symplectically on a symplectic manifold  $(P, \Omega)$  and  $H$  be an invariant Hamiltonian. Let  $z \in P$  be a fixed point for the corresponding Hamiltonian vector field and assume:*

**H 1** *the Hessian  $d^2H(z)$  is nondegenerate; and*

**H 2**  *$d^2H(z_0)$  restricted to a resonance subspace  $V_\lambda$  is positive definite. ( $V_\lambda$  is the subspace of  $T_zP$  that is the real part of the direct sum of all the generalized eigenspaces of eigenvalues of  $L = DX_H(z)$  that are multiples of the purely imaginary eigenvalue  $\lambda$ .)*

*Then for every isotropy subgroup  $\Sigma$  of the  $G \times S^1$  action on  $V_\lambda$ , and for  $\epsilon > 0$  sufficiently small, there are at least*

$$\frac{1}{2} \dim \text{Fix}(\Sigma, V_\lambda) \quad (3.5)$$

*periodic trajectories of  $X_H$  with periods near  $2\pi/|\lambda|$  and symmetry group containing  $\Sigma$ , on the energy surface  $H = E_0 + \epsilon$ .*

To apply this theorem, we need information about the eigenvalues and generalized eigenspace of the linearization  $DX_H(z)$  of the Hamiltonian vector field at the stable equilibrium. The Hamiltonian is quite complicated and so it is tedious to find the eigenvalues and eigenspace directly. Fortunately, we do not have to do this. Instead, we will fully exploit the  $Z_2$ -symmetry and solve using general facts about symplectic representations (see Guillemin and Sternberg [1984]). We identify the tangent space to  $P_\mu$  at the stable equilibrium with  $\mathbb{C}^2$  by setting

$$z_1 = \dot{\phi} + iv_1 \quad \text{and} \quad z_2 = \dot{\psi} + iv_2 . \quad (3.6)$$

Then the induced  $\mathbb{Z}_2$ -representation on this tangent space is decomposed into irreducible representations;  $\mathbb{C}^2 = \mathbb{C}_0 \oplus \mathbb{C}_1$  where  $\mathbb{C}_0$  is the trivial piece and  $\mathbb{C}_1$  is the non-trivial piece. In fact,  $\mathbb{C}_0 = \{(z_1, z_2) \mid z_1 + z_2 = 0\}$ , which is the tangent space to  $F$  at the equilibrium and  $\mathbb{C}_1 = \{(z_1, z_2) \mid z_1 - z_2 = 0\}$ .

The group  $\mathbb{Z}_2$  acts on  $\mathbb{C}_0$  trivially and on  $\mathbb{C}_1$  by  $(z, z) \mapsto (-z, -z)$ . Since Hessian of  $H$  at this equilibrium is positive definite, all eigenvalues of the linearization of  $X_H$  are imaginary and come in pairs  $\pm i\lambda_1, \pm i\lambda_2$  where  $\lambda_1$  and  $\lambda_2$  may be the same. Since  $\mathbb{C}_0$  is the tangent space to  $F$ , the linearization will have  $\mathbb{C}_0$  as a generalized eigenspace corresponding to, say,  $\pm i\lambda_1$ . Since we know that each generalized eigenspace is symplectic and pairwise orthogonal,  $\mathbb{C}_1$  will be the generalized eigenspace of  $\pm i\lambda_2$ .

We summarize the above discussions:

**Proposition 3.6** *The tangent space at the stable equilibrium identified with  $\mathbb{C}^2$  by (3.6) are decomposed into irreducible pieces of the induced representation of  $\mathbb{Z}_2$ :*

$$\mathbb{C}^2 = \mathbb{C}_0 \oplus \mathbb{C}_1$$

where

$$\mathbb{C}_0 = \{(z_1, z_2) \mid z_1 + z_2 = 0\} \quad \text{and} \quad \mathbb{C}_1 = \{(z_1, z_2) \mid z_1 - z_2 = 0\} . \quad (3.7)$$

Moreover, these irreducible components correspond to generalized eigenspaces of the linearization of  $X_H$  with the eigenvalues  $\pm i\lambda_1, \pm i\lambda_2$ , respectively,  $\lambda_1, \lambda_2 > 0$ .

From this proposition we conclude that the flows of the linearization  $L$  on  $\mathbb{C}_0$  and  $\mathbb{C}_1$  are equivalent to

$$\text{multiplication by } e^{2\pi i s \lambda_1} \text{ on } \mathbb{C}_0 \text{ and multiplication by } e^{2\pi i s \lambda_2} \text{ on } \mathbb{C}_1 . \quad (3.8)$$

Thus, we have  $\mathbb{Z}_2 \times S^1$  actions on  $\mathbb{C}_0$  and  $\mathbb{C}_1$ , respectively.

Next, we find the isotropy groups of these actions on each of  $\mathbb{C}_0$  and  $\mathbb{C}_1$ :

On  $\mathbb{C}_0$  the isotropy group is  $\mathbb{Z}_2 \times \{1\}$  and whole space  $\mathbb{C}_0$  is the fixed point space of real dimension 2.

On  $\mathbb{C}_1$  the isotropy group is  $\{-1\} \times \{-1\}$  and again the whole space  $\mathbb{C}_1$  is the fixed point space of real dimension 2.

Therefore, we have the following refinement of Theorem 3.1:

**Theorem 3.7** For any small  $\varepsilon > 0$ , we have at least one periodic orbit with  $\mathbb{Z}_2$ -symmetry and at least one periodic orbit with  $\{-1\} \times \{-1\}$  symmetry on the energy surface  $H^{-1}(E_0 + \varepsilon)$  in  $P_\mu$ .

**Remarks 1** As we mentioned before, these periodic orbits give quasi-periodic orbits in the original phase space  $T^*(S^1 \times S^1 \times S^1)$ . They have the pictures in Figure 7 in the stick representation viewed from a rotating frame, i.e., up to the diagonal  $S^1$ -action.

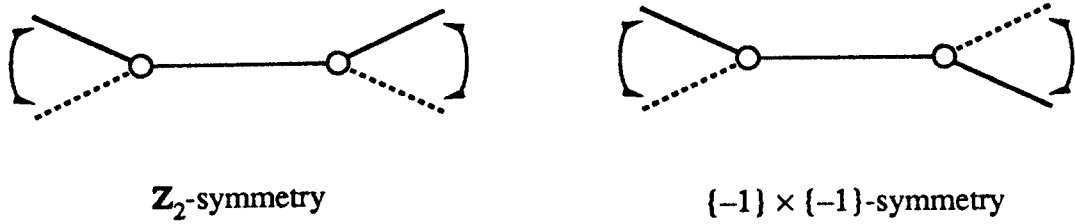


Figure 7

2 When  $\lambda_1 \neq \lambda_2$ , one can apply an equivariant version of the Liapunov Center Theorem to produce smooth families of periodic orbits with corresponding symmetries bifurcating from the stable equilibrium.

3 We conjecture that these periodic solutions are related to travelling waves for many bodies and in the continuum limit. ♦

Finally in this section, we examine some aspects of stability. If we let  $\phi_t$  be the Hamiltonian flow, then the *Floquet operator*  $M(u)$  of a periodic orbit  $u(t)$  with period  $T$  is defined by

$$M(u) = D\phi_T(u(0)) : T_{u(0)}P \rightarrow T_{u(0)}P . \quad (3.9)$$

If  $M(u)$  has all eigenvalues on the unit circle, then  $u$  is called *spectrally stable*. Note that  $M(u)$  always has a generalized eigenspace of dimension at least 2 with eigenvalue  $|1|$  because

$$u'(T) = D\phi_T(u(0)) \cdot u'(0) = u'(0) . \quad (3.10)$$

Now, let  $u_1$  and  $u_2$  be the periodic solutions in Theorem 3.7 whose periods  $T_1$  and  $T_2$  are near  $2\pi/|\lambda_1|$  and  $2\pi/|\lambda_2|$ , respectively. Then we have the following result about the spectral stability.

**Theorem 3.8** *If  $\lambda_1$  and  $\lambda_2$  are non-resonant, then the two periodic orbits which were found in Theorem 3.7 are spectrally stable if  $\epsilon > 0$  is small.*

**Proof** Note that the Floquet operator  $M(u_i)$  is close to  $\exp(-T_i DX_H(z))$  as  $\epsilon \rightarrow 0$ . Note that  $T_i \approx 2\pi/|\lambda_i|$  and  $\exp((-2\pi/|\lambda_i|)DX_H(z))$  has eigenvalue whose corresponding generalized eigenspace is of dimension 2 and so has one simple eigenvalue pair which lies on the unit circle. By the general rigidity of the behavior of the eigenvalues of perturbations of a symplectic matrix, we conclude that the eigenvalues of  $M(u_i)$  stay on the unit circle if  $\epsilon > 0$  is small. ■

## §4 Chaotic Solutions

In this section we show that the dynamics of the three coupled rigid body system is not integrable, having chaotic solutions of horseshoe type. This is done using the Holmes and Marsden [1983] version of Melnikov's method to perturb a homoclinic orbit in a problem with  $S^1$  symmetry. There are several homoclinic orbits one can use to perturb; a pair was described in Theorem 3.4. Here we perturb the two body problem by adding a third body near the center of mass of the second.

We first need to derive an expression for the Hamiltonian that is written so the perturbing terms are isolated. Refer to Figure 8.

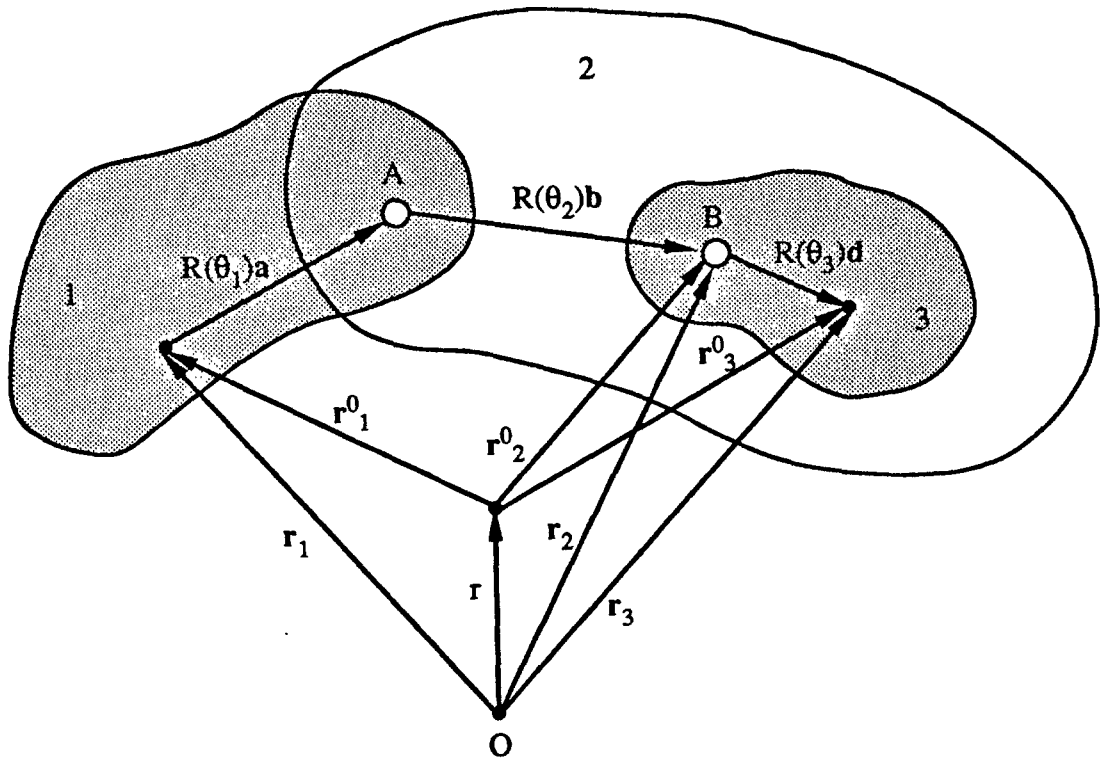


Figure 8

Let:

$O$  be the hinge point of bodies 1 and 2;

$B$  be the hinge point of bodies 2 and 3, and also the center of mass of body 2;

$a, b, d$  be the vectors between the centers of mass and hinge points of bodies 1, 2, 3 in the reference configuration;

$R(\theta)$  be the rotation through angle  $\theta$ ;

$r$  be the vector from  $O$  to the system center of mass;

$\mathbf{r}_1^0, \mathbf{r}_2^0, \mathbf{r}_3^0$  be vectors from the system center of mass to the body centers of mass;

$\theta_1, \theta_2, \theta_3$  be rotation angles from a reference configuration to the current configuration;

$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$  be position vectors for points in bodies 1, 2, 3 in the reference configuration;

$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$  be position vectors for points in bodies 1, 2, 3 in the current configuration.

As in [I], we have

$$\mathbf{x}_i = \mathbf{R}(\theta_i)\mathbf{X}_i + \mathbf{r}_i, \quad (4.1a)$$

$$\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{R}(\theta_1)\mathbf{a} + \mathbf{R}(\theta_2)\mathbf{b}, \quad (4.1b)$$

$$\mathbf{r}_3 = \mathbf{r}_2 + \mathbf{R}(\theta_3)\mathbf{d}, \quad (4.1c)$$

$$m\mathbf{r} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3. \quad (4.1d)$$

We compute the total kinetic energy as in [I] as

$$\begin{aligned} H = & \sum_{i=1}^3 \frac{1}{2} \text{trace}(\omega_i \mathbf{I}_i^T \omega_i) + \frac{p^2}{2m} \\ & + \frac{1}{2} m_1 \left\| -\frac{m_2 + m_3}{m} (\dot{\mathbf{R}}_1 \mathbf{a} + \dot{\mathbf{R}}_2 \mathbf{b}) - \frac{m_3}{m} \dot{\mathbf{R}}_3 \mathbf{d} \right\|^2 \\ & + \frac{1}{2} m_2 \left\| \frac{m_1}{m} (\dot{\mathbf{R}}_1 \mathbf{a} + \dot{\mathbf{R}}_2 \mathbf{b}) - \frac{m_3}{m} \dot{\mathbf{R}}_3 \mathbf{d} \right\|^2 \\ & + \frac{1}{2} m_3 \left\| \frac{m_1}{m} (\dot{\mathbf{R}}_1 \mathbf{a} + \dot{\mathbf{R}}_2 \mathbf{b}) - \frac{m_1 + m_3}{m} \dot{\mathbf{R}}_3 \mathbf{d} \right\|^2. \end{aligned} \quad (4.2)$$

Assume  $p = 0$ , without loss of generality, and that the reference configuration is chosen so  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{d}$  are parallel. Then we can write

$$H = H_d = H_0 + dH_1 + O(d^2) \quad (4.3)$$

where  $d = \|\mathbf{d}\|$  is our small parameter,

$$H_0 = \frac{1}{2} (I_1 + \epsilon a^2) \omega_1^2 + \frac{1}{2} (I_2 + \epsilon b^2) \omega_2^2 + \epsilon ab \cos \phi \omega_1 \omega_2 + \frac{1}{2} I_3 \omega_3^2, \quad (4.4a)$$



and

$$H_1 = \gamma(a \cos(\phi + \psi) \omega_1 \omega_3 + b \cos \psi \omega_2 \omega_3) \quad (4.4b)$$

where

$$\varepsilon = \frac{m_3(m_1 + m_2)}{m}, \quad \gamma = \frac{m_1 m_3}{m}, \quad \text{and} \quad O(d^2) = \frac{m_3(m_1 + m_2)}{m} d^2 \omega_3^2.$$

We can rewrite  $H$  as  $H = \frac{1}{2} \langle \omega, J_d \omega \rangle$  where  $\omega = (\omega_1, \omega_2, \omega_3)^T$ ,

$$J_d = \begin{pmatrix} \tilde{I}_1 & \varepsilon ab \cos \phi & \gamma ad \cos(\phi + \psi) \\ \varepsilon ab \cos \phi & \tilde{I}_2 & \gamma bd \cos \psi \\ \gamma ad \cos(\phi + \psi) & \gamma bd \cos \psi & \tilde{I}_3 \end{pmatrix} \quad (4.5)$$

and

$$\tilde{I}_1 = I_1 + \varepsilon a^2, \quad \tilde{I}_2 = I_2 + \varepsilon b^2, \quad \tilde{I}_3 = I_3 + \frac{m_3(m_1 + m_2)}{m} d^2.$$

Write  $J_d = J_0 + dJ_1 + O(d^2)$  where

$$J_0 = \begin{pmatrix} \tilde{I}_1 & \varepsilon ab \cos \phi & 0 \\ \varepsilon ab \cos \phi & \tilde{I}_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

and

$$J_1 = \begin{pmatrix} 0 & 0 & \gamma a \cos(\phi + \psi) \\ 0 & 0 & \gamma b \cos \psi \\ \gamma a \cos(\phi + \psi) & \gamma b \cos \psi & 0 \end{pmatrix}.$$

We need to write the kinetic energy with respect to the momentum  $\mu$  rather than the angular velocity. This is done using  $\mu = J_d \omega$ :

$$H_d(\mu) = \frac{1}{2} \langle \mu, J_d^{-1} \mu \rangle = H_0(\mu) + dH_1(\mu) + O(d^2)$$

where

$$H_1(\mu) = \left. \frac{\partial H_d}{\partial d} \right|_{d=0} = - \langle \mu, J_0^{-1} \left. \frac{\partial J}{\partial d} \right|_{d=0} J_0^{-1} \mu \rangle = - \langle \mu, J_0^{-1} J_1 J_0^{-1} \mu \rangle . \quad (4.6)$$

Here,

$$J_0^{-1} = \begin{pmatrix} \frac{1}{\Delta} \begin{pmatrix} \tilde{I}_2 & -\epsilon ab \cos \phi \\ -\epsilon ab \cos \phi & \tilde{I}_1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & I_3^{-1} \end{pmatrix}$$

(where  $\Delta = \tilde{I}_1 \tilde{I}_2 - \epsilon^2 a^2 b^2 \cos^2 \theta$ ), and so

$$J_0^{-1} J_1 J_0^{-1} =$$

$$\begin{pmatrix} 0 & 0 & \frac{\gamma I_3^{-1}}{\Delta} (\tilde{I}_2 a \cos(\phi + \psi) - \epsilon b^2 a \cos \phi \cos \psi) \\ 0 & 0 & \frac{\gamma I_3^{-1}}{\Delta} [(-\epsilon a^2 b \cos(\phi + \psi) \cdot \cos \phi) + \tilde{I}_1 b \cos \psi] \\ \frac{\gamma I_3^{-1}}{\Delta} (\tilde{I}_2 a \cos(\phi + \psi) - \epsilon b^2 a \cos \phi \cos \psi) & \frac{\gamma I_3^{-1}}{\Delta} [(-\epsilon a^2 b \cos(\phi + \psi) \cdot \cos \phi) + \tilde{I}_1 b \cos \psi] & 0 \end{pmatrix} .$$

Therefore,

$$H_0 = \frac{1}{2\Delta} (\tilde{I}_2 \mu_1^2 + \tilde{I}_1 \mu_2^2 - 2\epsilon ab \cos \phi \mu_1 \mu_2) + \frac{1}{2} I_3^{-1} \mu_3^2 \quad (4.7a)$$

and

$$H_1 = -\frac{\gamma I_3^{-1} a}{\Delta} \mu_1 \mu_3 (\tilde{I}_2 \cos(\phi + \psi) - \epsilon b^2 \cos \phi \cos \psi) - \frac{\gamma I_3^{-1} b}{\Delta} \mu_2 \mu_3 (-\epsilon a^2 \cos(\phi + \psi) \cos \phi + \tilde{I}_1 \cos \psi) . \quad (4.7b)$$

Now, notice that when  $d = 0$ , i.e.,  $B$  coincides with the center of mass of body 3, the system is completely integrable and we know that the reduced system has two homoclinic orbits, given in Figure 1 of [I]. This system is in the framework of the Melnikov method with an  $S^1$  symmetry as generalized by Holmes and Marsden [1983].

From (4.7a) we see that the unperturbed Hamiltonian  $H_0$  has an additional  $S^1$  symmetry given by  $\psi$ -rotations; or, equivalently, by  $\theta_3$ -rotations in the original system, which induces the obvious Poisson action whose moment mapping is exactly  $\mu_3 : T^*(S^1 \times S^1 \times S^1)/S^1 \rightarrow \mathbb{R}$ . Note that  $\mu_1 + \mu_2 + \mu_3$  is the moment mapping corresponding to the simultaneous uniform rotation; i.e., the diagonal action of  $S^1$ . The  $H_d$  and  $H_0$ -flows restrict to the symplectic leaf  $P_M = \{\mu_1, \mu_2, \mu_3, \phi, \psi \mid \mu_1 + \mu_2 + \mu_3 = M\}$ , and  $(\psi, \mu_3)$  are conjugate variables in this symplectic leaf. Note that the equations of motion for  $H_0$  are given by

$$\begin{aligned} \dot{\mu}_1 &= \frac{\partial H_0}{\partial \phi} , \quad \dot{\mu}_2 = -\frac{\partial H_0}{\partial \phi} + \frac{\partial H_0}{\partial \psi} , \quad \dot{\mu}_3 = -\frac{\partial H_0}{\partial \psi} , \\ \dot{\phi} &= \frac{\partial H_0}{\partial \mu_2} - \frac{\partial H_0}{\partial \mu_1} , \quad \dot{\psi} = \frac{\partial H_0}{\partial \mu_3} - \frac{\partial H_0}{\partial \mu_2} . \end{aligned} \quad (4.8)$$

By regrouping equations (4.8), we see that  $(\mu_1, \mu_2, \phi)$  can be separated; after solving this system we can substitute back to get the equations for  $(\mu_3, \psi)$ . Since  $\psi$  is the cyclic variable for  $H_0$ ,

$$\begin{aligned} \dot{\mu}_3 &= 0 , \quad \dot{\psi} = I_3^{-1} \mu_3 - \frac{1}{\Delta} (\tilde{I}_1 \mu_2 - \epsilon a b \cos \phi \mu_1) \quad (= \omega_3 - \omega_2) \\ &\equiv \Omega(t) . \end{aligned} \quad (4.9)$$

Let  $x(t) = (\mu_1(t), \mu_2(t), \phi(t))$  be a homoclinic orbit for the  $(\mu_1, \mu_2, \phi)$ -dynamics in  $\mu_1 + \mu_2 + \mu_3 \equiv M$ ,  $\mu_3 \equiv J$  where  $M, J$  are given constants.

As in Holmes and Marsden [1983], if we set

$$\psi(t) = \int_0^t \Omega(s) ds + \psi_0 ,$$

we have only to prove that the *Melnikov function*

$$M(\psi_0) = \int_{-\infty}^{\infty} \left\{ H_0, \frac{H_1}{\Omega} \right\} \left( x(t), \int_0^t \Omega(s) ds + \psi_0, J \right) dt \quad (4.10)$$

has simple zeroes to get "horseshoes", where  $\{ , \}$  is the bracket in the variables  $(\mu_1, \mu_2, \phi)$  in  $\{\mu_1 + \mu_2 + \mu_3 = M, \mu_3 = J\}$ .

Note that  $\Omega$  is an explicit function depending on  $\mu_1, \mu_2$ , and so will not be a constant as time changes. For reasons that will become clear, we will consider, instead of the function  $M(\psi_0)$ , the function

$$N(\psi_0, \mu_0) := M(\psi_0) \cdot \frac{1}{\gamma \mu_3 / I_2} ;$$

i.e.,

$$- \int_{-\infty}^{\infty} \left\{ H_0, \frac{H_1}{\Omega \gamma \mu_3 / I_3} \right\} \left( x(t), \int_0^t \Omega(s) ds + \psi_0, J \right) dt . \quad (4.11)$$

Now,

$$H_1' \equiv \frac{H_1}{\gamma \mu_3 / I_3} = - \frac{1}{\Delta} \{ a(\tilde{I}_2 \mu_1 - \epsilon ab \mu_2 \cos \phi) \cos(\phi + \psi) + b \cos \psi (\tilde{I}_1 \mu_2 - \mu_1 ab \epsilon \cos \phi) \} .$$

From now on, we will drop  $\tilde{\phantom{x}}$  on  $\tilde{I}_1$  and  $\tilde{I}_2$ , remembering  $\tilde{I}_1 \geq \epsilon a^2$  and  $\tilde{I}_2 \geq \epsilon b^2$ . Let us first consider  $N(\psi_0, 0)$ . Then  $\Omega(t) = -(I_1 \mu_2 - \epsilon ab \cos \phi \mu_1) / \Delta$ , and

$$\begin{aligned} \frac{H_1'}{\Omega} &= \left\{ b \cos \psi + a \cos(\phi + \psi) \frac{I_2 \mu_1 - \epsilon ab \mu_2 \cos \phi}{I_1 \mu_2 - \epsilon ab \mu_1 \cos \phi} \right\} \\ &= \left\{ b \left( \cos \int_0^t \Omega(s) ds \cdot \cos \psi_0 - \sin \int_0^t \Omega(s) ds \cdot \sin \psi_0 \right) \right. \\ &\quad \left. + a \frac{I_2 \mu_1 - \epsilon ab \mu_1 \cos \psi}{I_2 \mu_2 - \epsilon ab \mu_1 \cos \psi} \cos \left( \phi + \int_0^t \Omega(s) ds \right) \cos \psi_0 \right. \\ &\quad \left. - \sin \left( \phi + \int_0^t \Omega(s) ds \right) \sin \psi \right\} . \quad (4.12) \end{aligned}$$

We can assume without loss of generality that  $\mu_1$  and  $\mu_2$  are even functions and  $\phi$  is an odd

function; then  $\Omega$  is an even function and so

$$\int_0^t \Omega(s) ds$$

is an odd function of  $t$ . Thus

$$\begin{aligned} \left\{ H_0, \frac{H_1'}{\Omega} \right\} &= \left\{ H_0, b \cos \int_0^t \Omega(s) ds + a \frac{I_2 \mu_1 - \epsilon ab \mu_2 \cos \phi}{I_1 \mu_2 - \epsilon ab \mu_1 \cos \phi} \cos \left( \phi + \int_0^t \Omega(s) ds \right) \right\} \cos \psi_0 \\ &\quad - \left\{ H_0, b \sin \int_0^t \Omega(s) ds + a \frac{I_2 \mu_1 - \epsilon ab \mu_2 \cos \phi}{I_1 \mu_2 - \epsilon ab \mu_1 \cos \phi} \sin \left( \phi + \int_0^t \Omega(s) ds \right) \right\} \sin \psi_0 \\ &= \left\{ H_0, a \frac{I_2 \mu_1 - \epsilon ab \mu_2 \cos \phi}{I_1 \mu_2 - \epsilon ab \mu_1 \cos \phi} \cos \left( \phi + \int_0^t \Omega(s) ds \right) \right\} \cos \psi_0 \\ &\quad - \left\{ H_0, a \frac{I_2 \mu_1 - \epsilon ab \mu_2 \cos \phi}{I_1 \mu_2 - \epsilon ab \mu_1 \cos \phi} \sin \left( \phi + \int_0^t \Omega(s) ds \right) \right\} \sin \psi_0 . \end{aligned} \quad (4.13)$$

By symmetry, the first term of (4.13) will vanish after integration. Thus,

$$N(\psi_0, 0) = - \int_{-\infty}^{\infty} \left\{ H_0, a \frac{I_2 \mu_1 - \epsilon ab \mu_2 \cos \phi}{I_1 \mu_2 - \epsilon ab \mu_1 \cos \phi} \sin \left( \phi + \int_0^t \Omega(s) ds \right) \right\} dt \sin \phi_0 . \quad (4.14)$$

Thus, what we have to do is prove that

$$\int_{-\infty}^{\infty} \left\{ H_0, a \frac{I_2 \mu_1 - \epsilon ab \mu_2 \cos \phi}{I_1 \mu_2 - \epsilon ab \mu_1 \cos \phi} \sin \left( \phi + \int_0^t \Omega(s) ds \right) \right\} dt \neq 0 .$$

Assume, for example, that  $I_1 = I_2 = I$  and  $a = b = 1$ . Then the integrand becomes

$$\begin{aligned} &\left\{ H_0, \frac{I \mu_1 - \epsilon \mu_2 \cos \phi}{I \mu_2 - \epsilon \mu_1 \cos \phi} \sin \left( \phi + \int_0^t \Omega(s) ds \right) \right\} \\ &= \dot{\phi} \frac{\partial}{\partial \phi} \left\{ \frac{I \mu_1 - \epsilon \mu_2 \cos \phi}{I \mu_2 - \epsilon \mu_1 \cos \phi} \sin \left( \phi + \int_0^t \Omega(s) ds \right) \right\} \\ &\quad - \dot{\mu}_1 \left( \frac{\partial}{\partial \mu_2} - \frac{\partial}{\partial \mu_1} \right) \left\{ \frac{I \mu_1 - \epsilon \mu_2 \cos \phi}{I \mu_2 - \epsilon \mu_1 \cos \phi} \sin \left( \phi + \int_0^t \Omega(s) ds \right) \right\} . \end{aligned} \quad (4.15)$$

Here

$$\begin{aligned} \frac{\partial}{\partial \phi} \{ \} &= \frac{\epsilon I \sin \phi (\mu_2^2 - \mu_1^2)}{(I\mu_2 - \epsilon\mu_1 \cos \phi)^2} \sin \left( \phi + \int_0^t \Omega(s) ds \right) \\ &+ \frac{I\mu_1 - \epsilon\mu_2 \cos \phi}{I\mu_2 - \epsilon\mu_1 \cos \phi} \cos \left( \phi + \int_0^t \Omega(s) ds \right) \end{aligned} \quad (4.16)$$

and

$$\left( \frac{\partial}{\partial \mu_2} - \frac{\partial}{\partial \mu_1} \right) \{ \} = - \frac{(I^2 - \epsilon^2 \cos^2 \phi)(\mu_1 + \mu_2)}{(I\mu_2 - \epsilon\mu_1 \cos \phi)^2} \sin \left( \phi + \int_0^t \Omega(s) ds \right). \quad (4.17)$$

Let us express  $\mu_1, \mu_2$  as functions of  $\phi$  for one of the homoclinic orbits which is in the  $(\mu_1, \mu_2, \phi)$ -dynamics. It is given as the intersection of  $\mu_1 + \mu_2 = M - J$  and

$$H_0 = \frac{1}{2\Delta} (I\mu_1^2 + I\mu_2^2 - 2\epsilon \cos \phi \mu_1 \mu_2) = \frac{1}{4} \frac{(M - J)^2}{I - \epsilon} \quad (4.18)$$

(the energy of the homoclinic orbit).

Set  $v = \mu_2 > \mu_1$ . Then we can write

$$H_0 = \frac{1}{4} \left\{ \frac{(M - J)^2}{I + \epsilon \cos \phi} + \frac{v^2}{I - \epsilon \cos \phi} \right\}. \quad (4.19)$$

When

$$H_0 = \frac{1}{4} \left( \frac{(M - J)^2}{I - \epsilon} \right),$$

we get

$$v = \pm \sqrt{\Gamma} (M - J) \text{ where } \Gamma = \frac{\beta}{1 - \beta} (1 + \cos \phi) \frac{1 - \beta \cos \phi}{1 + \beta \cos \phi} \text{ and } \beta = \frac{\epsilon}{I}. \quad (4.20)$$

Consider the homoclinic orbit obtained from  $v = \sqrt{\Gamma} (M - J)$ . Then

$$\mu_2 = \frac{1 + \sqrt{\Gamma}}{2} (M - J) \quad \text{and} \quad \mu_1 = \frac{1 - \sqrt{\Gamma}}{2} (M - J) . \quad (4.21)$$

Substitute (4.21) into (4.16) and (4.17) to get

$$\begin{aligned} \frac{\partial}{\partial \phi} \{ , \} = & \frac{4 \beta \sin \phi \sqrt{\Gamma}}{\{(1 - \beta \cos \phi) + \sqrt{\Gamma} (1 + \beta \cos \phi)\}^2} \sin \left( \phi + \int_0^t \Omega(s) ds \right) \\ & + \frac{(1 - \beta \cos \phi) - \sqrt{\Gamma} (1 + \beta \cos \phi)}{(1 - \beta \cos \phi) + \sqrt{\Gamma} (1 + \beta \cos \phi)} \cos \left( \phi + \int_0^t \Omega(s) ds \right) , \end{aligned} \quad (4.22)$$

$$\left( \frac{\partial}{\partial \mu_2} - \frac{\partial}{\partial \mu_1} \right) \{ , \} = - \frac{4(1 - \beta^2 \cos^2 \phi)(M - J)^{-1}}{\{(1 - \beta \cos \phi) + \sqrt{\Gamma} (1 + \beta \cos \phi)\}^2} \sin \left( \phi + \int_0^t \Omega(s) ds \right) . \quad (4.23)$$

And

$$\dot{\mu}_1 = \frac{d\mu_1}{d\phi} \cdot \dot{\phi} = - \frac{1}{4} (M - J) \cdot \frac{\Gamma'}{\sqrt{\Gamma}} \dot{\phi} .$$

Thus  $N(\psi_0, 0) = A_1 + A_2 + A_3$ , where

$$A_1 = \int_{-\infty}^{\infty} \frac{-4\beta \sin \phi \sqrt{\Gamma} \dot{\phi}}{((1 - \beta \cos \phi) + \sqrt{\Gamma} (1 + \beta \cos \phi))^2} \sin \left( \phi + \int_0^t \Omega(s) ds \right) dt , \quad (4.24a)$$

$$A_2 = - \int_{-\infty}^{\infty} \frac{(1 - \beta \cos \phi) - \sqrt{\Gamma} (1 + \beta \cos \phi)}{(1 - \beta \cos \phi) + \sqrt{\Gamma} (1 + \beta \cos \phi)} \dot{\phi} \cos \left( \phi + \int_0^t \Omega(s) ds \right) dt , \quad (4.24b)$$

$$A_3 = \int_{-\infty}^{\infty} \frac{1 - \beta^2 \cos^2 \phi}{((1 - \beta \cos \phi) + \sqrt{\Gamma} (1 + \beta \cos \phi))^2} \frac{\Gamma'}{\sqrt{\Gamma}} \dot{\phi} \sin \left( \phi + \int_0^t \Omega(s) ds \right) dt . \quad (4.24c)$$

Now, note that

$$\Omega = - \frac{1}{\Delta} (I\mu_2 - \epsilon \cos \phi \mu_1) = - \frac{I\mu_2 - \epsilon \cos \phi \mu_1}{I^2 - \epsilon^2 \cos^2 \phi} = - \frac{1}{I} \cdot \frac{\mu_2 - \beta \cos \phi \mu_1}{1 - \beta^2 \cos^2 \phi} \quad (4.25)$$

and that all the integrand decays exponentially as  $t \rightarrow \pm \infty$ . Therefore all the above integrals are analytic functions with respect to  $1/I$  and  $\beta$ ,  $0 < \beta < 1$ ,  $0 < 1/I$ , and are continuous in the range

$0 < \beta < 1$ ,  $0 \leq 1/I$ . We will consider the limiting case when  $I = \infty$ . Then  $\Omega \equiv 0$  and so (4.24) becomes

$$A_1 = \int_{-\infty}^{\infty} \frac{-4\beta \sin \sqrt{\Gamma} \dot{\phi}}{((1 - \beta \cos \phi) + \sqrt{\Gamma} (1 + \beta \cos \phi))^2} \sin \phi \, dt, \quad (4.26a)$$

$$A_2 = - \int_{-\infty}^{\infty} \frac{(1 - \beta \cos \phi) - \sqrt{\Gamma} (1 + \beta \cos \phi)}{(1 - \beta \cos \phi) + \sqrt{\Gamma} (1 + \beta \cos \phi)} \dot{\phi} \cos \phi \, dt, \quad (4.26)$$

$$A_3 = \int_{-\infty}^{\infty} \frac{1 - \beta^2 \cos^2 \phi}{((1 - \beta \cos \phi) + \sqrt{\Gamma} (1 + \beta \cos \phi))^2} \frac{\Gamma'}{\sqrt{\Gamma}} \dot{\phi} \sin \phi \, dt. \quad (4.26c)$$

Changing variables, these become

$$A_1 = \int_{-\pi}^{\pi} \frac{-4\beta \sin^2 \phi \sqrt{\Gamma} \, d\phi}{((1 - \beta \cos \phi) + \sqrt{\Gamma} (1 + \beta \cos \phi))^2}, \quad (4.27a)$$

$$A_2 = - \int_{-\pi}^{\pi} \frac{(1 - \beta \cos \phi) - \sqrt{\Gamma} (1 + \beta \cos \phi)}{(1 - \beta \cos \phi) + \sqrt{\Gamma} (1 + \beta \cos \phi)} \cos \phi \, d\phi, \quad (4.27b)$$

$$A_3 = \int_{-\pi}^{\pi} \frac{1 - \beta^2 \cos^2 \phi}{((1 - \beta \cos \phi) + \sqrt{\Gamma} (1 + \beta \cos \phi))^2} \frac{\Gamma'}{\sqrt{\Gamma}} \sin \phi \, d\phi. \quad (4.27c)$$

It appears to be difficult to directly check whether the sum  $A_1 + A_2 + A_3$  is not equal to zero. To deal with this, let us compare the order of  $A_1$ ,  $A_2$ , and  $A_3$  as  $\sqrt{\beta} \rightarrow 0$ . The order of  $A_1$  is  $O(\beta^{3/2})$ . For  $A_2$  and  $A_3$ , set  $\sqrt{\beta} = \delta$  to get

$$A_2 = - \int_{-\pi}^{\pi} \frac{(1 - \delta^2 \cos \phi) - \frac{\delta}{\sqrt{1 - \delta^2}} \sqrt{(1 + \cos \phi)(1 - \delta^4 \cos^2 \phi)}}{(1 - \delta^2 \cos \phi) + \frac{\delta}{\sqrt{1 - \delta^2}} \sqrt{(1 + \cos \phi)(1 - \delta^4 \cos^2 \phi)}} \cos \phi \, d\phi. \quad (4.28)$$



$$= - \int_{-\pi}^{\pi} \frac{1 - \frac{\delta}{\sqrt{1-\delta^2}} \sqrt{\frac{(1+\cos\phi)(1+\delta^2\cos\phi)}{1-\delta^2\cos\phi}}}{1 + \frac{\delta}{\sqrt{1-\delta^2}} \sqrt{\frac{(1\cos\phi)(1+\delta^2\cos\phi)}{1-\delta^2\cos\phi}}} \cos\phi \, d\phi . \quad (4.29)$$

The integrand is equal to

$$\begin{aligned} & \frac{1 - \delta \left(1 + \frac{1}{2} \delta^2 + O(\delta^4)\right) (1 + \delta^2 \cos\phi + O(\delta^4)) \sqrt{1 + \cos\phi}}{1 + \delta \left(1 + \frac{1}{2} \delta^2 + O(\delta^4)\right) (1 + \delta^2 \cos\phi + O(\delta^4)) \sqrt{1 + \cos\phi}} \cos\phi \\ &= \left(1 - \left(\delta + \frac{1}{2} \delta^3 (1 + 2 \cos\phi) + O(\delta^5)\right) \sqrt{1 + \cos\phi}\right) \\ & \quad \times \left\{ \left(1 - \delta \left(1 + \frac{1}{2} \delta^2 (1 + 2 \cos\phi) + O(\delta^4)\right) \sqrt{1 + \cos\phi}\right) \right. \\ & \quad \left. + \frac{1}{2} (1 + \cos\phi) \delta^2 \left(1 + \frac{1}{2} \delta^2 (1 + 2 \cos\phi) + O(\delta^4)\right)^2 \right. \\ & \quad \left. + \frac{1}{6} (1 + \cos\phi)^{3/2} \delta^3 \left(1 + \frac{1}{2} \delta^2 (1 + 2 \cos\phi) + O(\delta^4)\right)^3 \right\} \cos\phi + O(\delta^4) \\ &= \cos\phi \left\{ 1 - 2\sqrt{1 + \cos\phi} \delta + \frac{3}{2} (1 + \cos\phi) \delta^2 \right\} + O(\delta^3) . \quad (4.30) \end{aligned}$$

Now check the coefficients of  $\delta$  and  $\delta^2$  in (4.29); the coefficient of  $\delta$  is equal to

$$2 \int_{-\pi}^{\pi} \sqrt{1 + \cos\phi} \cos\phi \, d\phi = \frac{8}{3} \sqrt{2} .$$

For  $A_3$ ,

$$\Gamma' = \frac{\beta}{1 - \beta} \cdot \frac{-\sin\phi(1 - \beta^2 \cos^2\phi) + 2\beta \sin\phi \cos\phi (1 + \cos\phi)}{(1 + \beta \cos\phi)^2} .$$

Thus the coefficient of  $\delta = \sqrt{\beta}$  in the expansion of  $A_3$  is computed as

$$\int_{-\pi}^{\pi} \frac{-\sin^2 \phi}{\sqrt{1 + \cos \phi}} d\phi = -\frac{8}{3} \sqrt{2} .$$

Unfortunately, the first order term in  $\delta$  in (4.29) vanishes, so we must check the coefficient of  $\delta^2$ . From (4.30), the coefficient of  $\delta^2$  in  $A_2$  equals

$$-\frac{3}{2} \int_{-\pi}^{\pi} \cos \phi (1 + \cos \phi) d\phi = -6\pi .$$

Similarly, the coefficient of  $\delta^2$  in the expansion of  $A_3$  is computed to be

$$\int_{-\pi}^{\pi} 2 \sin^2 \phi d\phi = 4\pi .$$

Thus the coefficient of  $\delta^2$  in  $A_1 + A_2 + A_3$  is given by  $-6\pi + 4\pi = -2\pi \neq 0$ . Therefore the coefficient of  $\delta^2$  in the expansion of  $N(\psi_0, 0)$  with respect to  $\beta$  when  $I \rightarrow \infty$  is not equal to zero. From this, we can conclude that the Melnikov integral has only simple zeros for generic parameter values if the distance between the hinge points and the center of mass of the third body is small. We summarize:

**Theorem 4.1** *If the distance between the center of mass of the third body and its hinge point is sufficiently small, then apart from isolated values of the system parameters, the dynamics of the three body system has Poincaré-Birkhoff-Smale horseshoes, so is non-integrable.*

## §5 Discussion

In this paper we have developed a fairly complete picture of the dynamics of the planar 3 body system. The kind of analysis we have presented enables one to study equilibria and their stability, bifurcations of equilibria, periodic and chaotic solutions. Computer graphics of the dynamics illustrating these features has been developed by Sreenath [1987].

While this analysis may be difficult to extend to a complex structure of  $n$  bodies, the detailed understanding of the dynamics of 3 bodies helps us to understand the relation between chaos, coherence, and stability in more complex structures and in the continuum limit  $n \rightarrow \infty$ . In fact, it is proved in a preprint of Y.-G. Oh that the straight - out configuration of the finite coupled rigid  $n$  - body system is always stable, but its continuum analogue turns out to not be formally stable. Moreover, we also have a good understanding of the special structure of periodic orbits bifurcated from the equilibrium for the symmetric coupled rigid body system.

We also point out that the detailed understanding of the Hamiltonian structure via symmetry reduction should assist in the development of numerical algorithms and the control theory for these systems.

In Grossman, Krishnaprasad, and Marsden [1987], the dynamics of coupled three dimensional rigid bodies is studied. The analysis there indicates that there may be a symmetric Hamiltonian Hopf bifurcation leading to interesting periodic and chaotic motions. Again one can conjecture the possibility of interesting three dimensional waves, such as helical waves, being built from an understanding of the few degrees of freedom situation. An eventual goal is to link this theory up with the infinite dimensional case in Krishnaprasad and Marsden [1987], Simo, Krishnaprasad, and Marsden [1987] and Krishnaprasad, Marsden, Posburgh, and Simo [1988].

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